Year 12 Semester 1 Examination, 2016

Question/Answer Booklet

Hale School

MATHEMATICS SPECIALIST

Section One
Calculator-free

Student Name

Teacher: (circle) Mr Hill Mr Lau

Score: (out of 53)

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	11	11	100	99	65
			Total	152	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **(53 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(8 marks)**

(a) Simplify $\left(\sqrt{3}-i\right)^6$ (3 marks)

$$\left(2cis(-\frac{\pi}{6})\right)^6$$
$$=\left(2^6 cis(-\pi)\right)$$
$$=-64$$

Convert to Polar form ✓

Applies D'Moivres Theroem ✓

States the answer ✓

(b) If z_1 = 2 + 3i , z_2 = 9 + 7i and $z_1 w = z_2$, find the complex number w in the form $a+b$i. (2 marks)

$$w=\frac{9+7i}{2+3i}$$
$$=\frac{9+7i}{2+3i}\times\frac{2-3i}{2-3i}$$
$$=\frac{39-13i}{13}$$
$$=3-i$$

Multiplies by Conjugate ✓

Correct Answer ✓

(c) z_1 and z_2 are roots of the equation $x^2+ax+b=0$, where a and b are real numbers. Given $z_1=2+\sqrt{3}$ i , state z_2 and the value of the constants a and b. (3 marks)

$$z_2=2-\sqrt{3}i$$

$$\left(x-\left(2+\sqrt{3}i\right)\right)\left(x-\left(2-\sqrt{3}i\right)\right)$$
$$=x^2-\left(2+\sqrt{3}i+2-\sqrt{3}i\right)x+\left(2+\sqrt{3}i\right)\left(2-\sqrt{3}i\right)$$
$$=x^2-4x+7$$

$$a=-4,\ b=7$$

Complex Conjugate ✓

Expands complex factors ✓

States correct values ✓

Question 2 **(7 marks)**

Consider the following system of equations:

$$2x + 3y - z = 1$$
$$x + 2y + 3z = 12$$
$$-x + y + (a-1)z = 3a$$

(a) Solve the system of equations when $a = 3$. (4 marks)

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & 2 & 3 & 12 \\ -1 & 1 & 2 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -1 & -7 & -23 \\ 0 & 3 & 5 & 21 \end{bmatrix} \begin{matrix} \\ R_1 - 2R_2 \\ R_2 + R_3 \end{matrix}$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -1 & -7 & -23 \\ 0 & 0 & -16 & -48 \end{bmatrix} \begin{matrix} \\ \\ 3R_2 + R_3 \end{matrix}$$

$$-16z = -48$$
$$z = 3$$
$$y = 2$$
$$x = -1$$

Substitutes a=3	✓
Correctly reduces rows	✓
Determines value for z	✓
Correct x and y	✓

(b) Determine the value(s) of a for which the system has infinite solutions. (3 marks)

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & 2 & 3 & 12 \\ -1 & 1 & (a-1) & 3a \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -1 & -7 & -23 \\ 0 & 3 & a+2 & 3a+12 \end{bmatrix} \begin{matrix} \\ R_1 - 2R_2 \\ R_2 + R_3 \end{matrix}$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -1 & -7 & -23 \\ 0 & 0 & a-19 & 3a-57 \end{bmatrix} \begin{matrix} \\ \\ 3R_2 + R_3 \end{matrix}$$

$$a = 19$$

Correctly reduces rows	✓✓
Identifies a=19 creates infinite solutions	✓

Question 3 **(7 marks)**

(a) Use the factor theorem to show that the complex number 3i is a root of
$z^4 - z^3 + 3z^2 - 9z - 54 = 0$ (2 marks)

$$(3i)^4 - (3i)^3 + 3(3i)^2 - 9(3i) - 54$$
$$= 81 + 27i - 27 - 27i - 54$$
$$= 0$$

Substitutes (3i) ✓
=0 ✓

(b) One of the solutions to the equation $z^3 = k$ is $z_1 = 2cis\frac{\pi}{6}$.

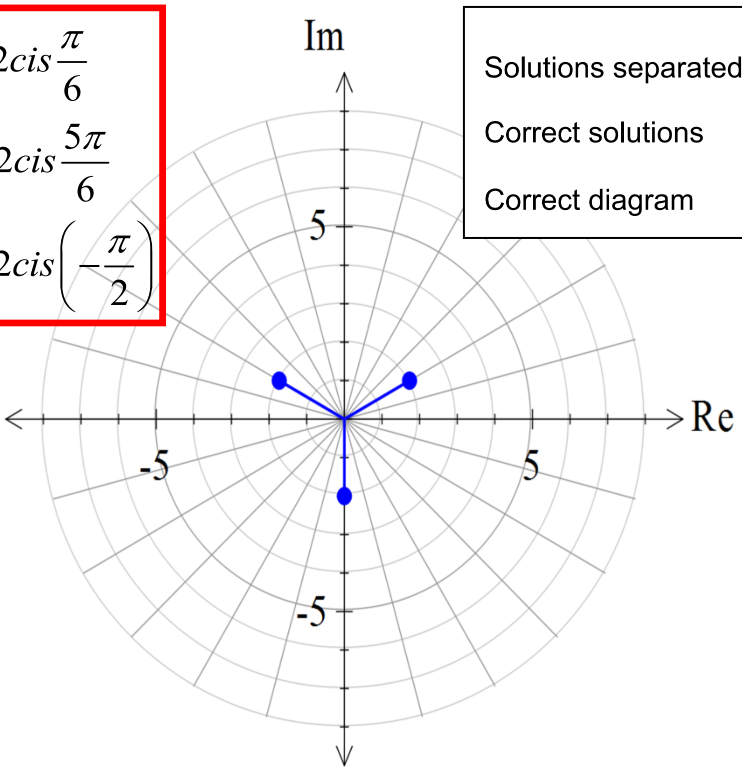
(i) Find the complex constant k in cartesian form. (2 marks)

$$k = \left(2cis\frac{\pi}{6}\right)^3 = 8cis\frac{\pi}{2} = 8i$$

Uses De-moivre's Theorem ✓
Correct k value ✓

(ii) On an Argand diagram, show the vectors representing all three solutions (z_1, z_2 and z_3) of $z^3 = k$ and give the solutions in polar form. (3 marks)

$$z_1 = 2cis\frac{\pi}{6}$$
$$z_2 = 2cis\frac{5\pi}{6}$$
$$z_3 = 2cis\left(-\frac{\pi}{2}\right)$$



Solutions separated by $\frac{2\pi}{3}$ ✓
Correct solutions ✓
Correct diagram ✓

Question 4 **(13 marks)**

Let $g(x)=\dfrac{3}{1-x}$, $h(x)=\sqrt{1-x}$ and $f(g(x))=-\dfrac{3}{x+2}$.

(a) State the largest possible domain of $goh(x)$. (2 marks)

$$D_x=\{0<x\leq 1\cup x<0 : x\in R\}$$

✓ ✓

(b) State the largest possible range of $goh(x)$. (2 marks)

$$R_y=\{y<0\cup 3\leq y : y\in R\}$$

✓ ✓

(c) Determine $g^{-1}(x)$. (2 marks)

$$x=\frac{3}{1-y}$$
$$(1-y)x=3$$
$$x-yx=3$$
$$y=\frac{x-3}{x}$$
$$g^{-1}(x)=\frac{x-3}{x}$$

Interchanges variables ✓
Correct inverse ✓

(d) Show that $gog^{-1}(x)=x$ (2 marks)

$$g(g^{-1}(x))=\frac{3}{1-\frac{x-3}{x}}$$
$$g(g^{-1}(x))=\frac{3x}{x-x+3}$$
$$g(g^{-1}(x))=\frac{3x}{3}$$
$$g(g^{-1}(x))=x$$

Substitutes inverse into g ✓
Multiplies by x/x ✓

Question 4 continued…

(e) Hence or otherwise, determine $f(x)$ (3 marks)

$$f\left(g(g^{-1}(x))\right) = -\frac{3}{\frac{x-3}{x}+2}$$

$$f(x) = -\frac{3x}{x-3+2x}$$

$$f(x) = \frac{x}{1-x}$$

Uses $f\left(g(g^{-1}(x))\right) = f(x)$ ✓

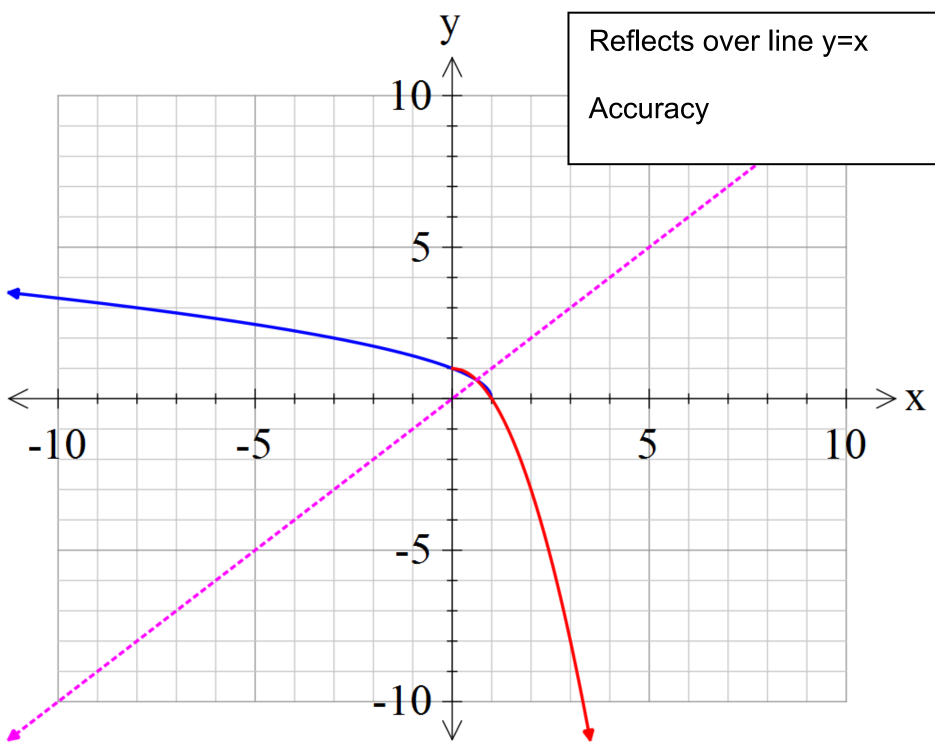
Multiplies by x/x ✓

States f(x) ✓

(f) Given the graph of $h(x)$ below, sketch the graph of $h^{-1}(x)$ (2 marks)

Reflects over line y=x ✓

Accuracy ✓



Question 5 **(5 marks)**

f is the function defined by $f(x) = \dfrac{(ax+b)^2}{x^2+cx+d}$ where a, b, c and d are constants.

The graph of f has poles at $x=-2$ and $x=-3$, a horizontal asymptote at $y=4$ and y-intercept at (0, 1.5).

Determine the possible values of a, b, c and d.

Poles at x=-2 and x=-3

$\therefore$ Denominator $= (x+2)(x+3) = x^2+5x+6$

$\therefore c=5, d=6$

At (0, 1.5)

$1.5 = \dfrac{b^2}{d}$

$9 = b^2$

$b = \pm 3$

$As\ x \to \infty, y \to 4$

$\therefore \dfrac{a^2}{1} = 4$

$a = \pm 2$

Uses poles to determine c, d	✓ ✓
Uses Intercept to determine b	✓
Uses asymptote to determine a	✓
+- values	✓

Question 6 **(9 marks)**

(a) Use De Moivre's Theorem to prove $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. (4 marks)

$$\text{Re}\left[\left(\cos 4\theta + i\sin 4\theta\right)\right] = \text{Re}\left[\left(\cos\theta + i\sin\theta\right)^4\right]$$

$$\cos 4\theta = \cos^4\theta + 6\cos^2\theta(i\sin\theta)^2 + (i\sin\theta)^4$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)(1-\cos^2\theta)$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Uses de-moivre's theorem ✓
Equates real parts ✓
Expands correctly ✓
Simplifies to correct expression ✓

(b) Hence, use this result to find real solutions to the equation $8x^4 - 8x^2 + 2 = 0$, giving the solutions in exact form. (5 marks)

$$8x^4 - 8x^2 + 1 = -1 \qquad \text{let } x = \cos\theta$$

$$8\cos^4\theta - 8\cos^2\theta + 1 = -1$$

$$\cos 4\theta = -1$$

$$4\theta = (2k+1)\pi \qquad k \in Z$$

$$\theta = \frac{(2k+1)\pi}{4}$$

$$x_0 = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$x_1 = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$x = \pm\frac{\sqrt{2}}{2}$$

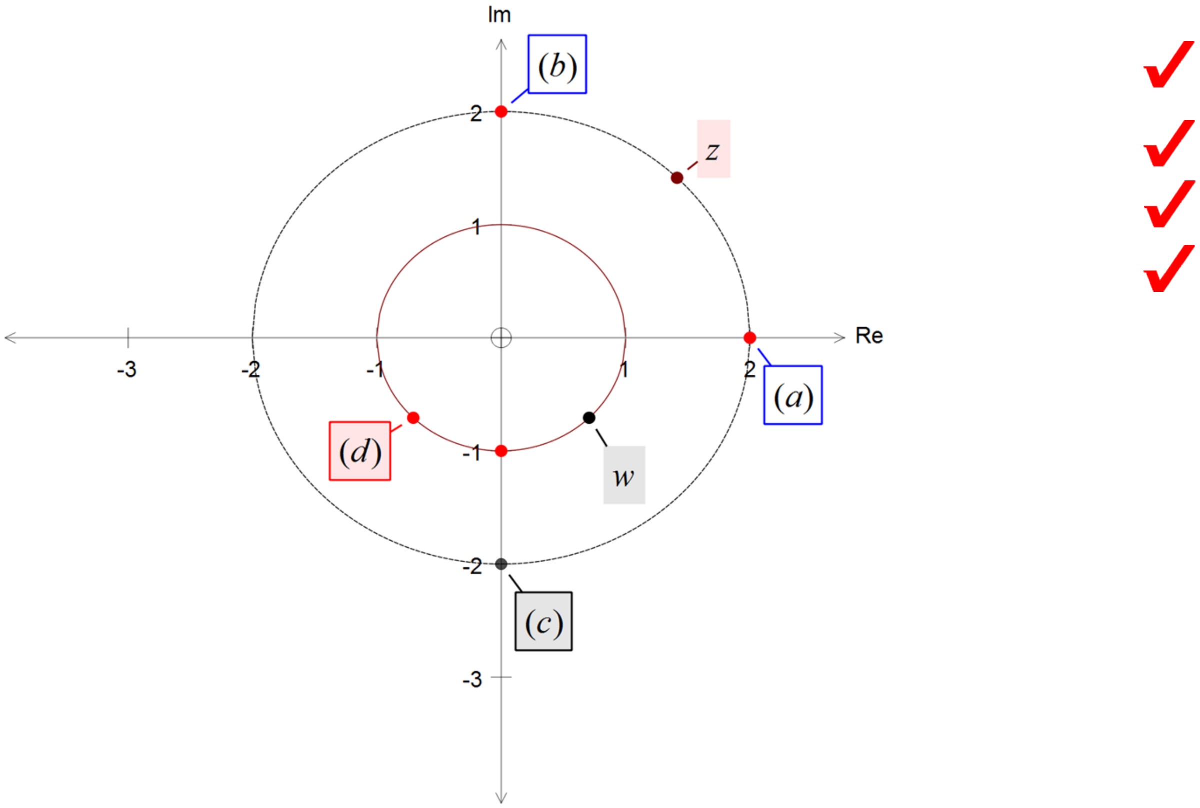
Splits equation ✓
Substitutes $x = \cos\theta$ ✓
Uses previous result ✓
Solves for θ ✓
Determines Solution ✓

Question 7 **(4 marks)**

The Argand diagram below shows the points representing the complex numbers w and z where $|w|=1$ and $|z|=2$. Plot on the same diagram, the points representing the complex numbers:

(a) $w \times z$ (b) $\overline{w} \times z$

(c) $\overline{(z / w)}$ (d) $\overline{w} - z$



END OF SECTION 1

Additional working space

Question number: _________

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